A presentation of Quantum Logic based on an *and then* connective *

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Abstract

When a physicist performs a quantic measurement, new information about the system at hand is gathered. This paper studies the logical properties of how this new information is combined with previous information. It presents Quantum Logic as a propositional logic under two connectives: negation and the *and then* operation that combines old and new information. The *and then* connective is neither commutative nor associative. Many properties of this logic are exhibited, and some small elegant subset is shown to imply all the properties considered. No independence or completeness result is claimed. Classical physical systems are exactly characterized by the commutativity, the associativity, or the monotonicity of the *and then* connective. Entailment is defined in this logic and can be proved to be a partial order. In orthomodular lattices, the operation proposed by Finch in [3] satisfies all the properties studied in this paper. All properties satisfied by Finch's operation in modular lattices are valid in Quantum Logic. It is not known whether all properties

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of Quantum Logic are satisfied by Finch's operation in modular lattices. Non-commutative, non-associative algebraic structures generalizing Boolean algebras are defined, ideals are characterized and a homomorphism theorem is proved. Keywords: Generalized Boolean Algebras, Non-associative Boolean Algebras, Non-commutative Boolean Algebras, Quantum Measurements, Measurement Algebras, Quantum Logic, Orthomodular lattices, Modular lattices. PACS: 02.10.-v.

1 Introduction

1.1 Background

Since its foundation in [1], an impressive amount of different systems have been proposed for Quantum Logic. This paper proposes a minimalistic syntax: one unary, $\neg$, and one binary, $*$, connectives. The binary connective is not the commutative and associative conjunction proposed by Birkhoff and von Neumann but the non-commutative, non-associative conjunction proposed by Finch in [3] that is interpreted in this paper as an *and then* connective acting on experimental propositions. The minimalistic syntax provides algebraic properties that have an immediate meaning for the logic of measurements in Quantum (and classical) Physics. Central properties of interest are properties of the binary connective, $*$, alone, that do not mention $\neg$. The algebraic structures, NCNAB-algebras, that correspond to this Quantum Logic are non-commutative, non-associative algebras that generalize Boolean algebras. The algebraic properties of the conjunction define an orthomodular partial order on the elements. The commutative NCNAB-algebras are exactly the Boolean algebras, fitting the accepted wisdom that Classical Physics is the special case of Quantum Physics one obtains when all observables commute.

This should be contradistincted with traditional presentations of Quantum Logic which:

- use a syntax including one unary connective and at least two binary connectives: conjunction, disjunction and often one or more implications,
- interpret conjunction as the (commutative) intersection of closed linear subspaces of Hilbert space, which is semantically problematic since the

projection on the intersection $A \cap B$ of two closed subspaces cannot be defined using the two projections on A and B,

- leads to a presentation in which the central properties considered such as distributivity, modularity or orthomodularity involve more than one connective, and have no obvious meaning for proof-theory.

Previous work on the non-commutative conjunction proposed by Finch [3], such as [8] have always considered this connective as defined in terms of more basic connectives. This paper is closely connected to [6]. The main difference is that, there, the basic operation was composition of projections and, here, the basic operation is the projection of one closed subspace on a closed subspace.

This paper leaves many questions unsolved.

1.2 Plan of this paper

In Section 2 the formal framework of Quantum Mechanics is presented and the representation of knowledge about a quantic system in this framework is discussed. Section 3 presents the syntax of the language that will be used to talk about quantic systems. Section 4 presents a semantic account of this language and defines Hilbert Space Quantic Logic. Section 5 defines the corresponding first-order structures, called NCNAB-algebras. They generalize Boolean algebras. It provides an in-depth study of NCNAB-algebras. Section 6 shows that orthomodular lattices, under Finch's [3] interpretation of the *and then* connective satisfy a list of central properties of NCNAB-algebras. All properties of even modular lattices, under this interpretation of the *and then* connective, hold in NCNAB-algebras. Section 7 studies ideals in NCNAB-algebras and proves a homomorphism theorem. Section 8 is a summary and conclusion.

2 What is a quantic proposition?

When, in [1], Birkhoff and von Neumann introduced Quantum Logic, they argued that an *experimental proposition* must be mathematically represented by a closed (linear) subspace of a Hilbert space. Let us develop this point.

The formalism generally accepted for Quantum Mechanics, brought to its final form by von Neumann in [9], considers the set of possible states of a

system to be the rays (i.e., one-dimensional subspaces) of a Hilbert space, say $\mathcal{H}$. A fundamental principle of Quantum Mechanics claims that if, from all one knows, the system could be in any one of two different states, then it could be in any one of the many different superpositions of those two states. Therefore propositions must be represented by linear subspaces of $\mathcal{H}$. Birkhoff and von Neumann argue that such subspaces must be closed.

Their argument is essentially the following. The basic pieces of information one can gather about a system are of the type: *the system is in the eigensubspace of some self-adjoint operator for some eigenvalue* λ. The eigensubspaces of any bounded linear operator are closed, and self-adjoint operators are bounded. Then they explain that the information one can gather about any system is built out of those basic pieces by intersection (for information given by different commuting operators) and linear sum (for different possible eigenvalues). They argue that, even for infinite such sums, the result has to be understood as the closure of the linear span of the closed subspaces considered.

If a proposition is represented by a closed subspace A, one may, at least in principle, test the system for this proposition. The measurement, represented by the projection on A will, if the system is in a state that satisfies the proposition (i.e., in A), give the corresponding eigenvalue with probability one and, if the system does not satisfy the proposition, the measurement will give, with some strictly positive probability, some other eigenvalue.

Consider now a totally unknown system on which one performs a sequence of two measurements. Before the first measurement, our knowledge is represented by the whole (closed) space $\mathcal{H}$. After the first measurement, our knowledge is represented by the closed subspace A that is the eigensubspace corresponding to the result obtained. After the second measurement, one knows, not only that the system is in the closed subspace B corresponding to the result obtained in the second measurement, but also that it is in the projection on B of some ray of A. We must therefore consider that, if A and B are meaningful closed subspaces, then the projection of A on B, i.e., the direct image of A under the transformation $\widehat{B}$, which is the projection on B, is a meaningful proposition. If one has performed a measurement whose result indicates A and, subsequently, one performs a measurement whose result indicates B, the knowledge that one possesses about the system is encapsulated in the subspace $\widehat{B}(A)$.

At this point, a very fundamental remark kicks in. The projection $\widehat{B}(A)$ of a closed subspace A on a closed subspace B is a subspace but is not always

closed. I am indebted to Semyon Alesker, Joseph Bernstein and Vitali Milman for enlightening me and providing me with an explicit counter-example. The counter-example is based on an unbounded operator whose graph is closed. By a result of Banach (1932) no such operator can be defined on the whole space, and one must build one such operator defined on only part of the space.

There is no way, then, we can consider an arbitrary Hilbert space $\mathcal{H}$ and the family of all closed subspaces of $\mathcal{H}$. We could decide to consider only those Hilbert spaces $\mathcal{H}$ for which the set of all closed subspaces is closed under projections, but there is absolutely no reason to stick to the idea, discussed critically by Birkhoff and von Neumann, that we should consider all closed subspaces of $\mathcal{H}$. It seems much more natural not to put restrictions on $\mathcal{H}$ but to consider only families of closed subspaces that are closed under projections. This is what will be done in Section 4.

3 Syntax

A syntax for denoting *measurements* and propositions to talk about them will be described now. Terms denote measurements.

Definition 1 *Let V be a denumerable set (of atomic terms). The set of* quantic terms *over V will be denoted by $QTerms(V)$ and is defined inductively by:*

1. *an element of V (an atomic term) is a quantic term,*
2. 1 *is a quantic term,*
3. *if x is a quantic term, then $\neg x$ is a quantic term,*
4. *if x and y are quantic terms, then $x * y$ is a quantic term, and*
5. *these are the only quantic terms.*

We shall write quantic terms using parentheses when useful and assuming that $\neg$ has precedence over $*$.

One could consider a more extreme minimalistic approach based on the following remark. If one reflects on the two expressions $(x * y) * z$ and

$x * (y * z)$, one notices that the former has an immediate experimental interpretation: the system may result from a measurement x followed by y followed by z. The latter expression does not present such a natural interpretation. Its meaning is that the system may result from a measurement of x and then a measurement that it could have been the case that y and then z were measured: a quite unnatural proposition to make, since it is not clear how one could measure that the system could have been in a state satisfying y and then z without measuring first y and then z. Therefore, one could have restricted the rule 4) above to: if x is a quantic term and y is a literal (i.e., atomic term or negation of an atomic term), then $x * y$ is a quantic term. This interesting possibility would probably be best treated in the framework of a calculus of sequents, and is left for future work.

Propositions talk about terms.

Definition 2 *A* simple *quantic proposition on V is a pair of elements of $QTerms(V)$, written $x = y$ for $x, y \in QTerms(V)$. The* conditional *quantic propositions on V are defined in the following way:*

1. *a simple quantic proposition is a conditional quantic proposition,*
2. *if $x = y$ is a simple quantic proposition and P is a* conditional *quantic proposition then* **if** x = y **then** P *is a conditional quantic proposition, and*
3. *these are the only conditional quantic propositions.*

Notation: ***The proposition*** **if w = x then if y = z then P** ***will be denoted:*** **if w = x and y = z then P*****. The simple proposition*** $x * y = x$ ***will denoted*** $x \leq y$***.***

In Section 4 we shall propose a semantics for the calculus of conditional quantic propositions, based on the geometry of Hilbert spaces.

4 Semantics

We shall formally define the families of closed subspaces we are interested in.

Definition 3 *Let $\mathcal{H}$ be a Hilbert space and M be a family of closed subspaces of $\mathcal{H}$. The family M is said to be* a P-family *iff*

- *$\mathcal{H} \in M$,*
- *for any $A \in M$, $A^{\perp} \in M$,*
- *for any $A, B \in M$, $\widehat{B}(A) \in M$.*

Set-theorists: note that we use the term *family* only for convenience since the families considered are sets. Note that, as mentioned in Section 1, the projection $\widehat{B}(A)$ is not always a closed subspace: M is a P-family only if such projections amongst members of the family are closed. There are many examples of P-families. For example, the set of all closed subspaces of a finite-dimensional Hilbert space is a P-family. For any Hilbert space $\mathcal{H}$, the family containing two elements: $\mathcal{H}$ and the null subspace is a P-family.

An interpretation f of $QTerms(V)$ into a P-family M of $\mathcal{H}$ associates with every quantic term an element of M such that:

- $f(1) = \mathcal{H}$,
- $f(\neg x) = f(x)^{\perp}$,
- $f(x * y) = \widehat{f(y)}(f(x))$.

Definition 4 *If $x = y$ is a simple quantic proposition over V, and f is an interpretation of $QTerms(V)$ into a P-family M, we shall say that $x = y$ is* satisfied *under f iff $f(x) = f(y)$. For a conditional quantic proposition* **if** x = y **then** P *we shall say that it is satisfied under f iff either P is satisfied under f or $x = y$ is not satisfied under f. A simple (resp. conditional) proposition is* valid in a P-family M *iff it is satisfied under any interpretation f into M. A simple (resp. conditional) proposition is* Hilbert-valid *iff it is valid in any P-family.*

The relation $\leq$ defined following Definition 2 is interpreted as subset inclusion.

Lemma 1 *Let f be an interpretation of $QTerms(V)$ into a P-family M. The simple proposition $x \leq y$ is satisfied under f iff $f(x) \subseteq f(y)$.*

Proof: Let the closed subspaces of $\mathcal{H}$, A and B be defined by: $A = f(x)$ and $B = f(y)$. We see that $A \subseteq B$ iff $\widehat{B}(A) = A$ iff $f(x * y) = f(x)$. ∎

Hilbert Space Quantic Logic is defined to be the set of all Hilbert-valid conditional propositions.

5 Non-Commutative, Non-Associative Boolean algebras

In this section, an effort is made to try and define the algebraic structures that can be taken as the essence of Quantum Logic. Three principles are guiding us:

- Language: we are looking for a family of general algebras whose type consists of two constants, a unary operation and a binary operation. Clearly other presentations may be considered, in a way that is similar to the many presentations of Boolean algebras. The only properties that we shall consider are properties that can be expressed as conditional propositions.
- Every P-family defines a structure in the family. This is a disputable assumption: one may think that not all P-families are meaningful for Quantum Mechanics and therefore that we may have to consider a subclass of P-families. In this paper only conditional propositions that are valid amongst all P-families will be considered.
- Every Boolean Algebra is an algebra of the family. This assumption is based on the strong feeling that Quantum Logic should not be seen as incompatible with classical logic, as is the case with the currently prevailing view of Quantum Logic, as attested by the results of Kochen and Specker, but that classical logic should be a special case of Quantum Logic. More precisely, classical logic is Commutative Quantum Logic (when for every x, y, $x * y = y * x$).

We consider structures $\langle M, 0, 1, \neg, * \rangle$ where M is a non-empty set, 0 and 1 are elements of M, $\neg$ is a unary function $M \longrightarrow M$ and $*$ is a binary function $M \times M \longrightarrow M$.

Definition 5 *A structure* $\langle M, 0, 1, \neg, * \rangle$ *is a* non-commutative, non-associative Boolean algebra *(NCNAB-algebra) iff it satisfies, for all interpretations of atomic terms in M, all conditional quantic propositions valid in Hilbert Space Quantum Logic.*

Note that Definition 5 does not require that 0 be different from 1.

It would be nice to be able to present now a list of conditional quantic propositions valid in Hilbert Space Quantum logic and show that any

structure satisfying those propositions is (isomorphic to) an NCNAB-algebra. This paper does not provide such a completeness result.

We shall present a number of conditional quantic propositions that are valid in Hilbert Space Quantum Logic and prove interesting properties for all structures that satisfy those properties, and therefore also for any NCNAB-algebra. No claim is made about the completeness of the list, and no claim is made about the independence of the properties listed in the sequel.

In Section 5.1, we shall present propositions that do not contain $\neg$. A first result claims that they are valid in Hilbert Space Quantum Logic. Its proof is postponed to Section 6. A second result shows that in any structure satisfying those propositions, the relation $\leq$ is a partial order. In Section 5.2, we shall present propositions that deal with $\neg$, claim that they are valid in Hilbert Space Quantum Logic (proof postponed) and show that any structure that satisfies those propositions and those of Section 5.1 and is commutative (or associative, or monotonic) is a Boolean algebra. In Section 5.3 we shall present valid propositions which, at this stage, cannot be proven to follow from the propositions of Sections 5.1 and 5.2. The reader should notice that all the propositions presented below have a natural flavor and represent ways of proving properties of quantic systems.

5.1 Properties of *and then*

Our first set of propositions deal with $*$ only. We shall say that x *and* y *commute* if $x * y = y * x$.

Theorem 1 *The following conditional quantic propositions are valid in Hilbert Space Quantum Logic.*

1. **Global Cautious Commutativity** *if* $x * y \leq x$ *then* $x * y = y * x$*,*
2. **Cautious Associativity** *if* $x * y = y * x$*, then, for any* $z \in M$*,* $z * (x * y) = (z * x) * y$*,*
3. **Local Cautious Commutativity** *if* $(z * x) * y \leq x$ *and* $(z * y) * x \leq y$*, then* $(z * x) * y = (z * y) * x$*,*
4. **Z** $0 * x = 0 = x * 0$*,*
5. **N** $1 * x = x = x * 1$*,*

6. **Left Monotony** *if* $x \leq y$*, then,* $x * z \leq y * z$*.*

Remarks:

- the binary operation $*$ is not assumed to be associative or commutative.

- Taking M to be a Boolean algebra, 0 to be the bottom element, 1 the top element, $\neg$ to be complementation and $*$ to be greatest lower bound, one obtains a model of all of the properties above, in which $*$ is associative and commutative, as well as a model of the properties of Theorems 3 and 6.

- **Global Cautious Commutativity** (GCC) is a weak commutativity property, it claims that, under certain circumstances, $*$ is commutative. The commutativity property asserted $x * y = y * x$ represents a global commutation property: x and y commute in any context. Commutation in a specific context z, a local commutation property, is expressed as $(z * x) * y = (z * y) * x$ and appears in the property of **Local Cautious Commutativity** (LCC) below. Theorem 2, item 9) shows that two propositions that commute globally, commute locally in any context.

- **Cautious Associativity** (CA) is a weak associativity property: under certain circumstances, i.e., if x and y commute, we have associativity for z, x and y.

- LCC is a weak commutativity property, it claims that, under certain circumstances, propositions x and y commute *locally*, i.e., in the context of z.

- Z expresses the fact that 0 is a zero for the operation $*$.

- N expresses the fact that 1 is a neutral element for the operation $*$.

- **Left Monotony** (LM) expresses the fact that the operation $*$ is monotone, with respect to $\leq$, in its left argument. A symmetric property of right monotony would imply commutativity since $x \leq 1$ would imply $y * x \leq y * 1 = y$ and GCC would then imply $x * y = y * x$.

Proof: One could prove directly, without much difficulty, that the properties of Theorem 1 are valid in Hilbert Space Logic. Since a stronger result, validity

in Orthomodular Logic, will be proved in Theorem 8, we postpone the proof. ∎

We may now prove that any structure satisfying the properties of Theorem 1 has many interesting properties.

Theorem 2 *The following properties hold in any structure that satisfies the properties GLC, CA, LCC, Z, N, and LM of Theorem 1:*

1. $0 \leq x \leq 1$,
2. $x * y \leq y$,
3. $x \leq x$, *i.e., the relation* $\leq$ *is reflexive, i.e.,* $x * x = x$,
4. *if* $x \leq y$ *then* x *and* y *commute,*
5. *the relation* $\leq$ *is antisymmetric,*
6. *the relation* $\leq$ *is transitive,*
7. *the relation* $\leq$ *is a partial order,*
8. *if* $x * y = y * x$*, then, for any* $z \in M$ *we have:* $z * (y * x) = z * (x * y) = (z * x) * y = (z * y) * x$,
9. *if* $x * y = y * x$*, then, for any* $z \in M$ *we have:* $(z * x) * y \leq x$ *(and* $(z * y) * x \leq y$*),*
10. *if* $x \leq y$*, then for any* $z \in M$*:* $z * x = (z * y) * x$.

Proof:

1. By Z and N.
2. By 1) above, $x \leq 1$. By LM, $x * y \leq 1 * y$. By N, $x * y \leq y$.
3. By 2) above, $1 * x \leq x$ and now, by N, $x \leq x$.
4. If $x * y = x$, by 3) of this Lemma, $x * y \leq x$ and, by (GCC), x and y commute.
5. Assume $x \leq y$ and $y \leq x$. By 4) above, x and y commute. But $x * y = x$ and $y * x = y$. We conclude that $x = y$.

6. Assume $x \leq y$ and $y \leq z$. We have $x = x * y = x * (y * z)$. But, by 4) above, y and z commute and, therefore, by CA we have $x * (y * z) = (x * y) * z = x * z$.

7. Obvious from the above.

8. From the assumption: $z * (y * x) = z * (x * y)$. By CA $(z * x) * y = z * (x * y)$ and also $(z * y) * x = z * (y * x)$.

9. By 8) above, and then 2) above, $(z * x) * y = z * (y * x) \leq y * x \leq x$. By 6) above, we conclude that $(z * x) * y \leq x$.

10. By 4) x and y commute and by 8) $(z * y) * x = z * (x * y) = z * x$.

∎

5.2 Properties of negation

We shall now deal with properties that involve both $*$ and $\neg$. We shall write $x \perp y$ for $x * y = 0$.

Theorem 3 *The following conditional quantic propositions are valid in Hilbert Space Quantum Logic.*

1. **NP** $x * \neg x = \neg x * x = 0$, *i.e.,* $x \perp \neg x$ *and* $\neg x \perp x$,

2. **RNL** *if* $x * z \leq y$ *and* $x * \neg z \leq y$, *then* $x \leq y$.

Remarks:

- NP, and RNL may be considered to be the proof rules that define negation. NP parallels a left introduction rule. RNL is a non-commutative left elimination rule.

- The property LNL, dual to RNL and expressed: if $z * x \leq y$ and $\neg z * x \leq y$, then $x \leq y$ is also valid in Hilbert Space Quantum Logic. It will be described and discussed in Section 5.3.

- The properties RNL and LNL are an important novelty of this paper. All the properties of Theorems 1 and 3, except RNL, are satisfied in Hilbert space when $*$ is interpreted as intersection and $\neg$ as orthogonal complement, the interpretation proposed by [1]. Neither RNL nor its

dual LNL are satisfied in this interpretation. Both are very natural rules that express a very basic rule of reasoning, reasoning by cases: to prove α it is enough to prove that α holds if β holds *and* that α holds if $\neg\beta$ holds. Such reasoning by cases is valid in classical logic. It is also valid in many (preferential) non-monotonic logics [5]. It is also used in Quantum Physics. The following presents a use of RNL. To prove that a system prepared in a certain way has a certain quantic property, it is enough to show that, after some measurement, all possible resulting systems have the property. Suppose, for example, that one prepares many copies of a quantic system and then measures, on each copy, its spin along some direction d'. One finds many possible values for the spin along the direction d'. If, then, on each of the resulting systems (with different values for the spin along d') one measures the value 0 for the spin along a direction d, this is a proof that the original system (before measuring along d') had a zero spin along d. Such a proof-rule seems to be crucially needed because, even if one measures the spin along d immediately (without measuring first along d') one cannot, in effect, exclude the possibility that some interaction between the system and its environment occured, resulting in some unknown measurement.

Proof: As for Theorem 1, the proof is postponed to Theorem 8. ∎

A series of theorems will now describe properties of all structures satisfying the properties above.

Theorem 4 *The following properties hold in any structure that satisfies the properties GLC, CA, LCC, Z, N, and LM of Theorem 1 and the properties NP and RNL of Theorem 3.*

1. $\neg(\neg x) = x$,
2. $0 = \neg 1$ *and* $1 = \neg 0$,
3. *the relation* $\perp$ *is symmetric,*
4. $x \leq y$ *iff* $x \perp \neg y$,
5. $x \leq y$ *iff* $\neg y \leq \neg x$,
6. *if* $x \leq y$ *and* $y \perp z$*, then* $x \perp z$,
7. *if* $y \leq x$ *and* $y \leq \neg x$ *then* $y = 0$,

8. *if* $x \leq y$ *and* $\neg x \leq y$, *then* $y = 1$,

9. *if* $x \leq y$ *and* $x \leq z$, *then* $x \leq y * z$,

Proof:

1. By Theorem 2, item 2) $\neg\neg x * x \leq x$. By NP and Theorem 2, item 1) $\neg\neg x * \neg x = 0 \leq x$. We conclude, by RNL, that $\neg\neg x \leq x$. Similarly we can show that $x \leq \neg\neg x$. We conclude, by Theorem 2, that $x = \neg\neg x$.

2. By NP $\neg 1 * 1 = 0$. By N $\neg 1 * 1 = \neg 1$. Therefore $\neg 1 = 0$ and, by 1) above, we have $\neg 1 = \neg\neg 0 = 0$.

3. if $x * y = 0$, then, by Theorem 2, item 1), $x * y \leq x$ and, by Theorem 2, item 4) x and y commute and therefore $y * x = 0$.

4. If $x \leq y$, we have $x * \neg y = (x * y) * \neg y$. But, by NP, y and $\neg y$ commute and therefore, by CA and then NP and Z, $x * \neg y = x * (y * \neg y) = x * 0 = 0$.

   If $x * \neg y = 0$, then $x * \neg y \leq y$ by Theorem 2, item 1). But $x * y \leq y$ by Theorem 2, item 2). We conclude, by RNL, that $x \leq y$.

5. $x \leq y$ iff, by 4), $x \perp \neg y$, iff, by 3), $\neg y \perp x$ iff, by 1), $\neg y \perp \neg\neg x$ iff, by 4), $\neg y \leq \neg x$.

6. If $y \perp z$, we have, by 1), $y \perp \neg\neg z$ and, by 6) $y \leq \neg z$. By transitivity of $\leq$ we have $x \leq \neg z$ and therefore $x \perp \neg\neg z$ and $x \perp z$.

7. $y \leq x$ implies $y * \neg x \leq 0$. $y \leq \neg x$ implies $y * \neg\neg x = 0$ and $y * x \leq 0$. By RNL, then, $y \leq 0$ and since $0 \leq y$, $y = 0$ by Theorem 2, item 5).

8. Assume $x \leq y$ and $\neg x \leq y$. By 5) above we have $\neg y \leq \neg x$ and $\neg y \leq \neg\neg x = x$ and, by 7), we have $\neg y = 0$, therefore $y = \neg\neg y = \neg 0 = 1$ by 2).

9. Assume $x \leq y$ and $x \leq z$. By LM, $x * z \leq y * z$. But, by 4) $x * \neg z = 0 \leq y * z$. By RNL, then, $x \leq y * z$.

▮

The next lemma deals with commuting propositions.

Lemma 2 *In any structure that satisfies the properties of Theorems 1 and 3:*

if all three propositions x, y and z commute pairwise, then x commutes with $y * z$,

2. *if x commutes with y, then x commutes with $\neg y$,*

3. *if x and y commute, then $x * y$ is their greatest lower bound and $\neg(\neg x * \neg y)$ their least upper bound,*

4. *if x and y commute, then $\neg(x * y) * y \leq \neg x$,*

5. **Robbins equation** *if x and y commute then $x = \neg(\neg(x * y) * \neg(x * \neg y))$,*

6. **Orthomodularity** *if $x \leq y$, then y is the least upper bound of x and $\neg x * y$.*

Proof:

1. By CA $x * (y * z) = (x * y) * z$ since y and z commute. Since x and y commute $(x * y) * z = (y * x) * z$. But x and z commute and, by Theorem 2, item 8) $(y * x) * z = (y * z) * x$.

2. Assume x and y commute. We have, by Z, NP and Theorem 2, item 8):

   $$0 = 0 * y = (\neg x * x) * y = (\neg x * y) * x.$$

   Therefore $\neg x * y \perp x$, $\neg x * y \perp \neg\neg x$, $\neg x * y \leq \neg x$ and, by GCC, $\neg x$ and y commute.

3. For arbitrary x and y, $x * y \leq y$ by Theorem 2, item 2); also $x * y$ is greater or equal to any lower bound of x and y, by Theorem 4, item 9). The fact that x and y commute gives us the last property needed: $x * y = y * x \leq x$ by Theorem 2, item 2).

   By 2 $\neg x$ and $\neg y$ commute. Therefore $\neg x * \neg y$ is the greatest lower bound of $\neg x$ and $\neg y$. By Theorem 4, item 5), $\neg(\neg x * \neg y)$ is therefore the least upper bound of $\neg\neg x$ and $\neg\neg y$.

4. This is property (4) of Finch [3], for the special case x and y commute. The claim holds without this assumption, see Theorem 6. Assume x

and y commute. By Theorem 2, item 2), we have $(\neg(x * y) * y) * \neg x \leq \neg x$. But, by CA, we have:

$$(\neg(x * y) * y) * x = \neg(x * y) * (y * x) = \neg(x * y) * (x * y) = 0 \leq \neg x.$$

By RNL we conclude that $\neg(x * y) * y \leq \neg x$.

5. It is enough to prove that, if x and y commute $\neg x = \neg(x * y) * \neg(x * \neg y)$. We have: $x * y = y * x \leq x$ by Theorem 2, item 2) and therefore, by Theorem 4, item 5) $\neg x \leq \neg(x * y)$. By 2) above, x commutes with $\neg y$ and $x * \neg y = \neg y * x \leq x$ by Theorem 2, item 2) and therefore, by Theorem 4, item 5) $\neg x \leq \neg(x * \neg y)$. By Theorem 4, item 9), we have $\neg x \leq \neg(x * y) * \neg(x * \neg y)$.

   Consider, now that, by 1) and 2) just above x, $\neg(x * y)$ and $\neg(x * \neg y)$ commute pairwise. By CA, then, we have

   $$(\neg(x * y) * \neg(x * \neg y)) * x = \neg(x * y) * (\neg(x * \neg y) * x)) \leq \neg(x * \neg y) * x \leq y$$

   by 4) above, but we also have

   $$(\neg(x * y) * \neg(x * \neg y)) * x = \neg(x * \neg y) * (\neg(x * y) * x) \leq \neg(x * y) * x) \leq \neg y$$

   by 4) above. By Theorem 4, item 7),

   $$(\neg(x * y) * \neg(x * \neg y)) * x = 0.$$

   Now, by Theorem 4, item 4), $(\neg(x * y) * \neg(x * \neg y)) \leq \neg x$.

6. If $x \leq y$ then clearly y is an upper bound for x and for $(\neg x) * y$ by Theorem 2, item 2). Suppose now that $x \leq z$ and $(\neg x) * y \leq z$. Since $x \leq y$, x and y commute, and, by just above, $y = \neg(\neg(y * x) * \neg(y * \neg x))$. Therefore $y = \neg(\neg x * \neg(y * \neg x))$. By 3 above, y is the least upper bound of x and $y * \neg x = \neg x * y$.

∎

Definition 6 *A structure is* commutative *iff for any $x, y \in M$, $x * y = y * x$. A structure is* associative *iff for any $x, y, z \in M$, $(x * y) * z = x * (y * z)$. A structure is* monotone *iff for any $x, y \in M$, $x * y \leq x$.*

Theorem 5 *For a structure $A = \langle M, 0, 1, \neg, *\rangle$ satisfying the properties of Theorems 1 and 3 the following propositions are equivalent:*

1. *A is associative,*
2. *A is monotone,*
3. *A is commutative,*
4. *A is a Boolean algebra.*

The failure of monotonicity is a hallmark of the approach to Quantum Logic taken in [2]. Theorem 5 shows that this failure is inherently linked to the failure of associativity and commutativity. It was the feeling of many that, since the hallmark of Quantum Mechanics, as opposed to Classical Mechanics, is the non-commutativity of operators, Quantum Logic should, in some way, be non-commutative. Theorem 5 shows why it also has to be non-associative, a property that is more surprising.

Proof: Assume A is associative. Consider arbitrary elements x and y. We shall show that $x * y \leq x$. By associativity: $(x * y) * \neg x = x * (y * \neg x)$. But, by Theorem 2, item 2), $y * \neg x \leq \neg x$ and, by Theorem 4, item 6) and 1): $y * \neg x \perp \neg\neg x = x$. Therefore, by Theorem 4, item 3) we have $x * (y * \neg x) = 0$ and $(x * y) * \neg x = 0$, $x * y \perp \neg x$ and, by Theorem 4, item 4) $x * y \leq x$.

If A is monotone, then, by GCC, it is commutative.

Assuming A is commutative, we could use any of many different characterizations of Boolean algebras to show that it is a Boolean algebra. We shall use the one conjectured by Robbins. McCune [7] proved Robbins conjecture: any structure in which $*$ is associative, commutative and satisfies the Robbins equation, for any elements x and y:

$$\neg(\neg x * y) * \neg(\neg x * \neg y) = x,$$

is a Boolean algebra. The operation $*$ is commutative by assumption. It is associative by CA. It satisfies the Robbins equation by Lemma 2, item 5).

A Boolean algebra is associative. ∎

Definition 7 *Let M be any NCNAB-algebra and let $X \subseteq M$ be a set of propositions of M. The sub-algebra generated by X, $M(X)$ is the smallest sub-algebra of M containing X.*

Note that $M(X)$ is an NCNAB-algebra since the intersection of a family of NCNAB-algebras is an NCNAB-algebra due to the conditional-equational form of the properties defining an NCNAB-algebra.

Lemma 3 *Let M be any NCNAB-algebra and let $X \subseteq M$ be a set of pairwise commuting propositions: i.e., for any $x, y \in X$ $x * y = y * x$, then the sub-algebra of M generated by X, $M(X)$ is a commutative NCNAB-algebra.*

Proof: By Lemma 2, items 1) and 2). ▮

5.3 Additional propositions valid in Hilbert Space Quantum Logic

Some additional propositions that are valid in Hilbert Space Quantum Logic will be presented here. The question whether these properties follow from those of Theorems 1 and 3 is still open.

Theorem 6 *The following properties hold in any NCNAB-algebra.*

1. **LNL** *if $z * x \leq y$ and $\neg z * x \leq y$, then $x \leq y$,*
2. **NN** *if $x \leq y$ and $x * \neg z \leq y$, then $x * z \leq y$,*
3. **F4** $y * (x * y)' \leq x'$.

LNL is the dual of RNL. NN is a paradoxical rule of proof: to prove y after one measures x and z, it is enough to prove . NN is a rule of cautious monotony and the converse of RNL. F4 is not easily interpreted in terms of quantic measurements. F4 is property (4) of Finch [3]. A special case was proved in Lemma 2, item 4.
Proof: The proof is postponed to Theorem 8. ▮

Lemma 4 *In any structure that satisfies the properties of Theorems 1 and 3 and F4, we have $x * (x * y)' \leq y'$.*

Proof: Since $x * y \leq y$, y and $(x * y)'$ commute. Therefore, $(x * (x * y)') * y = (x * y) * (x * y)' = 0 \leq y'$. But $(x * (x * y)') * y' \leq y'$ and we conclude that $x * (x * y)' \leq y'$. ▮

6 Orthomodular and Modular Quantum Logic

A different, weaker, semantics, based on orthocomplemented lattices may be considered. It was proposed by Finch in [3].

An interpretation f of $QTerms(V)$ into an orthocomplemented lattice $\langle X, \perp, \top, {}', \leq \rangle$ associates with every quantic term an element of M such that:

- $f(1) = \top$,
- $f(\neg x) = f(x)'$,
- $f(x * y) = (f(x) \vee f(y)') \wedge f(y)$.

Quantic propositions are given the obvious interpretation. Validity is defined as usual, for diferent families of orthocomplemented lattices: orthomodular, modular, and Boolean algebras. Orthomodular (resp. modular, Boolean) Quantum Logic is the set of all conditional propositions valid in orthomodular (resp. modular, Boolean) lattices. It is easy to see that in Boolean lattices, one has: $x * y = x \wedge y$ and therefore Boolean Quantum Logic is classical logic. But even in modular lattices $*$ is different from $\wedge$: consider the modular lattice of all subspaces of a Hilbert space.

Let us now sort out the relations between all those logics we considered: Hilbert Space Quantum Logic (HSQL), Orthomodular Quantum Logic (OQL), Modular Quantum Logic (MQL) and Boolean Logic (BL).

Theorem 7

$$OQL \subseteq MQL \subseteq HSQL \subset BL.$$

The rightmost inclusion is strict. It is not known whether OQL and HSQL are different.

In [1], Birkhoff and von Neumann proposed *modular* lattices as the structure of Quantum Logic. The research community did not chose this path and pursued the orthomodular path. Theorem 7 shows that, for the limited language considered in this paper, one may go the modular way.

Proof: Orthomodular Quantum Logic is a subset of Modular Quantum Logic since any modular lattice is orthomodular. We do not know whether the inclusion is strict. To see that Modular Quantum Logic is a subset of Hilbert Space Quantum Logic consider that any P-family is part of a modular lattice:

the lattice of all subspaces of $\mathcal{H}$. Complementation in the lattice is orthogonal complementation in Hilbert space. We are left to show that, in a P-family, the lattice operation defined by Finch is projection. In other terms, that given any two closed subspaces A and B of the P-family, the projection of A on B, $\widehat{B}(A)$, is $(A + B^{\perp}) \cap B$.

Lemma 5 *Let $\mathcal{H}$ be Hilbert. If A is any (not necessarily closed) linear subspace of $\mathcal{H}$ and B is any closed subspace of $\mathcal{H}$, then $\widehat{B}(A) = (A + B^{\perp}) \cap B$.*

Proof: $\vec{u} \in \widehat{B}(A)$ iff there is some $\vec{v} \in A$ such that $\vec{u} = \widehat{B}(\vec{v})$ iff $\vec{u} \in B$ and there is some $\vec{v} \in A$ such that $\vec{v} - \vec{u} \perp B$ iff $\vec{u} \in B$ and there are some $\vec{v} \in A$ and $\vec{w} \in B^{\perp}$ such that $\vec{u} = \vec{v} + \vec{w}$ iff $\vec{u} \in (A + B^{\perp}) \cap B$. ▮

It is not known whether Hilbert Space Quantum Logic is different from Modular Quantum Logic, or even whether it is different from Orthomodular Logic. The orthoarguesian law of [4] that traditionally separates Hilbert space logic from orthomodular logic is not obviously expressible in terms of $*$ and $\neg$ only.

Hilbert Space Quantum Logic is a strict subset of Boolean Logic. Indeed any Boolean Algebra is a field of subsets of some set X. Consider now the Hilbert space whose orthonormal basis is X. The elements of the field are closed subspaces and they form a P-family. MQL is therefore a subset of Boolean Logic. It is a strict subset since HSQL is not commutative. ▮

We shall now prove that all the properties of HSQL that were mentioned in Section 5 are part of OQL, the weakest of our logics, therefore proving Theorems 1, 3 and 6.

Let us assume an orthomodular lattice and define $a * b = (a \vee b') \wedge b$. First, note that the relation $\leq$ we define in NCNAB-algebras coincides with the ordering of the lattice. If we use $\leq$ to represent the order of the lattice: $x \leq y$ iff $x * y = x$. Proof: Assume $x \leq y$, then, by orthomodularity $x = y \wedge (x \vee y')$, i.e., $x = x * y$. Conversely, if $x = y \wedge (x \vee y')$, then $x \leq y$.

Lemma 6 *If $z * x \leq y$, then $z * x \leq z * (x \wedge y)$.*

Proof: By definition $z * x \leq z \vee x'$. Therefore $z * x \leq z \vee x' \vee y' = z \vee (x \wedge y)'$. But, by definition $z * x \leq x$ and, by assumption, $z * x \leq y$. We conclude that $z * x \leq (z \vee (x \wedge y)') \wedge x \wedge y = z * (x \wedge y)$. ▮

Lemma 7 *If $x \leq y$, then for any z, $z * x = (z * y) * x$.*

Proof: By orthonormality $z \leq (z \vee y') \wedge y \vee y'$. By assumption, $y' \leq x'$ and therefore $z \leq (z \vee y') \wedge y \vee x' = (z * y) \vee x'$. Therefore $z \vee x' \leq (z * y) \vee x'$ and $z * x \leq (z * y) * x$.

But $y' \leq x'$ implies: $z \vee y' \leq z \vee x'$, and $(z \vee y') \wedge y \leq z \vee x'$. Therefore $z * y \leq z \vee x'$, $((z * y) \vee x') \wedge x \leq (z \vee x') \wedge x$, i.e., $(z * y) * x \leq z * x$. ∎

Lemma 8 *If $(z * x) * y \leq x$, then $(z * x) * y = z * (x \wedge y)$.*

Proof: Assume $(z * x) * y \leq x$. We have $(z * x) * y \leq x \wedge y$ and $(z * x) * y = ((z * x) * y) * (x \wedge y)$. By Lemma 7, $((z * x) * y) * (x \wedge y) = (z * x) * (x \wedge y) = z * (x \wedge y)$. ∎

The next lemma shows that orthomodular structures satisfy some limited form of distributivity.

Lemma 9 *If $z' \leq x$ and $z' \leq y$ then $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. Therefore $(x \vee y) * z = (x * z) \vee (y * z)$.*

Proof: In any lattice and without any assumption $(x \vee y) \wedge z \geq (x \wedge z) \vee (y \wedge z)$.

If $z' \leq x$, we have, by orthomodularity, $x = z' \vee z \wedge x$. Similarly, $z' \leq y$ implies $y = z' \vee z \wedge y$. Therefore $(x \vee y) \wedge z = (z \wedge x \vee z \wedge y \vee z') \wedge z$. But $z \wedge x \vee z \wedge y \leq z$ and, by orthonormality: $z \wedge x \vee z \wedge y = z \wedge (z \wedge x \vee z \wedge y \vee z')$.

The last claim follows trivially. ∎

Lemma 10 $(x \vee y) \wedge (x \vee y') = x \vee (x \vee y') \wedge y$.

Proof: Without any hypothesis, in any lattice $x \vee (x \vee y') \wedge y \leq (x \vee y) \wedge (x \vee y')$.

By orthomodularity, it is now enough to show that we have:

$$(x \vee (x \vee y') \wedge y)' \wedge (x \vee y) \wedge (x \vee y') = 0,$$

i.e.,

$$x' \wedge (x' \wedge y \vee y') \wedge (x \vee y) \wedge (x \vee y') = 0.$$

But $(x' \wedge y \vee y') \wedge (x \vee y') = y'$ by orthomodularity since $y' \leq x \vee y'$. Therefore $x' \wedge (x' \wedge y \vee y') \wedge (x \vee y) \wedge (x \vee y') = x' \wedge y' \wedge (x \vee y) = 0$. ∎

Theorem 8 *Properties GCC, CA, LCC, Z, N, LM, NP, RNL, LNL and NN are valid in Orthomodular Quantic Logic and therefore in Hilbert Space Quantic Logic.*

Proof: Let us show now that GCC holds. Assume $x * y \leq x$. We shall show that $x * y = y * x$. First, note that, by Lemma 6, $x * y \leq x * (x \wedge y) = x \wedge y \leq (x \vee y') \wedge y$. Therefore $x * y \leq y * x$. By orthonormality, now, it is enough to prove that $(x * y)' \wedge (y * x) = 0$, i.e., $(x * y)' \wedge (y \vee x') \wedge x = 0$. But $(x * y)' \wedge x = x \wedge (x' \wedge y \vee y') \leq (x \vee y') \wedge (x' \wedge y \vee y' = y'$ by orthonormality, since $y' \leq x \vee y'$. Therefore $(x * y)' \wedge x \leq x \wedge y'$ and

$$(x * y)' \wedge (y \vee x') \wedge x \leq x \wedge y' \wedge (x' \vee y) = (x' \vee y)' \wedge (x' \vee y) = 0.$$

Let us show that CA holds. Assume $x * y = y * x$. We have $x * y \leq x \wedge y \leq x * y$. Therefore $x * y = x \wedge y$. We have $z * (x * y) = z * (x \wedge y)$. By Lemma 7, then, $z * (x * y) = (z * x) * (x \wedge y) = ((z * x) * y) * (x \wedge y)$. But $((z * x) * y \leq y$ and $((z * x) * y \leq x * y = y * x \leq x$ and therefore $((z * x) * y \leq x \wedge y$ and, as noticed above, $((z * x) * y) * (x \wedge y) = (z * x) * y$.

The property LCC follows directly from Lemma 8. Properties Z, N, LM and NP are obvious.

Let us show that RNL holds. Assume $x * z \leq y$ and $x * z' \leq y$. By orthomodularity: $x \leq ((x \vee z') \wedge z) \vee z'$ and therefore $x \leq y \vee z'$. Also $x \leq ((x \vee z) \wedge z') \vee z$ and therefore $x \leq y \vee z$. Therefore $x \leq (y \vee z) \wedge (y \vee z') \leq y$.

Let us show that LNL holds. By Lemma 9, $(z * x) \vee (z' * x) = 1 * x = x$. But, by assumption: $(z * x) \vee (z' * x) \leq y$.

Let us show that NN holds. By Lemma 10 $x \vee (x * z') = x \vee ((x \vee z) \wedge z') = (x \vee z') \wedge (x \vee z)$. We see that $x * z = (x \vee z') \wedge z \leq (x \vee z') \wedge (x \vee z) = x \vee (x * z')$.

Let us show that F4 holds. $y * (x * y)' = (y \vee (x * y)) \wedge (x * y)' = y \wedge (x * y)' = y \wedge ((x' \wedge y) \vee y'$. By Orthonormality this last expression is less or equal x'.
∎

7 Ideals and a homomorphism theorem

In this section, we generalize the notions of homomorphisms, kernels and ideals that are fundamental in the study of Boolean algebras. We prove a generalized homomorphism theorems: in non-commutative algebras kernels and ideals coincide.

Definition 8 *Let S_i, $i = 1, 2$ be structures of the type considered in Section 5 of carriers M_1 and M_2 respectively. A function $f : M_1 \longrightarrow M_2$ is a homomorphism from S_1 to S_2 iff, for any $x, y \in M_1$:*

1. $f(0) = 0$,

2. $f(1) = 1$,

3. $f(\neg x) = \neg f(x)$, *and*

4. $f(x * y) = f(x) * f(y)$.

Definition 9 *If S is a structure of carrier M, a binary relation $\sim$ on M is said to be a congruence relation iff:*

1. *$\sim$ is an equivalence relation,*

2. *if $x \sim y$ then $\neg x \sim \neg y$,*

3. *if $x_1 \sim x_2$ and $y_1 \sim y_2$ then $x_1 * y_1 \sim x_2 * y_2$.*

Any homomorphism f defines a congruence relation $\sim_f$ by $x \sim_f y$ iff $f(x) = f(y)$. The kernel of f, $Ker(f)$ is the equivalence class of 0. We shall now study the relation between kernels and congruences.

The notion of an ideal is key. We need the following definition.

Definition 10 *Assume M is the carrier of a structure and $I \subseteq M$. We shall define two binary relations on M:*

1. *$x \leq_I y$ iff $x * \neg y \in I$, and*

2. *$x \sim_I y$ iff $x \leq_I y$ and $y \leq_I x$.*

Definition 11 *Assume $S = \langle M, 0, 1, \neg, * \rangle$ is an NCNAB-algebra. A set $I \subseteq M$ is an* ideal *of S iff, for any $x, y, z \in M$:*

1. *$0 \in I$,*

2. *if $x \in I$ then, for any $y \in M$ $x * y \in I$ and $y * x \in I$,*

3. *for any $x, y, z \in M$, if $x * y \in I$ and $z * \neg y \in I$ then $x * z \in I$.*

4. *if $(x * y) \leq Ix$ then $x * y \sim_I y * x$,*

5. *if $x * y \sim_I y * x$, then for any $z \in M$, $(z * x) * y \sim_I z * (x * y)$,*

6. *if $(z * x) * y \leq_I x$ and $(z * y) * x \leq_I y$, then $(z * x) * y \sim_I (z * y) * x$,*

Condition 3 corresponds to the Boolean condition: if x and y are in I, then $x \vee y$ is in I: if x and y are in I, then $(x \vee y) \wedge x$ and $(x \vee y) \wedge \neg x$ are in I and therefore $x \vee y$ is in I. Conditions 4, 5 and 6 deal with the non-commutativity of $*$: they are trivially satisfied in a commutative structure.

Lemma 11 *Assume I is an ideal.*

1. *if $x \in I$ and $y \leq x$ then $y \in I$,*
2. *if $x * y \in I$, then $y * x \in I$.*

Proof:

1. By assumption and 2) we have $y * x \in I$. But $y * x = y$.
2. We have $y * \neg y = 0 \in I$ and $x * y \in I$. By 3) above $y * x \in I$.

▮

Lemma 12 *Let S_1 be an NCNAB-algebra and f is a homomorphism of domain $S1$, then, its kernel is an ideal.*

Proof:

1. By definition of a morphism $f(0) = 0$.
2. $f(x) = 0$ implies $f(x * y) = f(x) * f(y) = 0 * f(y) = 0$ and also $f(y * x) = f(y) * f(x) = f(y) * 0 = 0$. Note that $x \leq_I y$ iff $x * \neg y \in I$ iff $f(x) * \neg f(y) = 0$ iff $f(x) \leq f(y)$. Also $x \sim_I y$ iff $f(x) = f(y)$.
3. Assume $f(x) * f(y) \leq f(x)$. By GCC, $f(y) * f(x) = f(x) * f(y) \leq f(y)$ and our conclusions hold.
4. Assume $f(x) * f(y) = f(y) * f(x)$. By CA $(f(z) * f(x)) * f(y) = f(z) * (f(x) * f(y))$.
5. By LCC.
6. Assume $f(x) * f(y) = 0$ and $f(z) * \neg f(y) = 0$. We know that $f(x)$ and $f(y)$ commute and also that $f(z) \leq f(y)$. Therefore $f(x) * f(z) = (f(x) * f(y)) * f(z) = 0$.

▮

Lemma 13 *If I is an ideal of an NCNAB-algebra of carrier M, we have:*

1. *if $x \leq y$ then $x \leq_I y$,*
2. *the relation $\leq_I$ is transitive, and a quasi-order,*
3. *the relation $\sim_I$ is an equivalence relation,*
4. *if $x \leq_I y$ then $\neg y \leq_I \neg x$, and*
5. *if $x \leq_I y$ then, for any z $x * z \leq_I y * z$,*
6. *$x \leq_I y$ iff $x * y \sim_I x$,*
7. *if $x * y \sim_I y * x$ then $\neg x * y \sim_I y * \neg x$.*

Proof:

1. If $x \leq y$, then $x * \neg y = 0 \in I$.
2. Assume $x * \neg y \in I$ and $y * \neg z \in I$. By Lemma 11 we have $\neg z * y \in I$ and by Definition 11, item 3) $x * \neg z \in I$. Since the relation $\leq_I$ is clearly reflexive by item 1) above it is a quasi-order.
3. The relation is reflexive and transitive, by the above. It is symmetric by definition.
4. Assume $x * \neg y \in I$. Then, by Lemma 11 $\neg y * x \in I$ and $\neg y * \neg\neg x \in I$. Therefore $\neg y * \neg\neg x \in I$ and $\neg y \leq_I \neg x$.
5. Assume $x * \neg y \in I$. Since z and $\neg(y * z)$ commute, we have $(x * z) * \neg(y * z) = x * (z * \neg(y * z))$. But $z * \neg(y * z) \leq \neg y$. Therefore $(x * z) * \neg(y * z) = (x * \neg y) * (z * \neg(y * z)) \in I$.
6. Assume $x * y \sim_I x$. We have $x \leq_I x * y \leq y$. By parts 1) and 2) above we conclude that $x \leq_I y$. Assume $x \leq_I y$. We have $y * x \leq x \leq_I y$. By parts 1) and 2) above we conclude that $y * x \leq_I y$. By Definition 11, part 4) then $x * y \sim_I y * x$ and $x * y \leq_I x$. But $x \leq_I y$ and, by 5 above, $x * x \leq_I y * x \leq_I x * y$. We conclude that $x \leq_I x * y$.
7. Assume $x * y \sim_I y * x$. By Definition 11, part 6), $(\neg x * x) * y \sim_I (\neg x * y) * x$. We see that $(\neg x * y) * x \sim_I 0$, i.e., $(\neg x * y) * x \in I$, i.e., $\neg x * y \leq_I \neg x$. We conclude by Definition 11, part 4) that $\neg x * y \sim_I y * \neg x$.

▮

Theorem 9 *If I is an ideal of an NCNAB-algebra of carrier M, then the binary relation $\sim_I$ is a congruence relation.*

Proof: The relation $\sim_I$ is obviously symmetric. By Lemma 13 it is easily seen to be reflexive and transitive. It is therefore an equivalence relation. By Lemma 13, 4) $x \sim_I y$ implies $\neg x \sim_I \neg y$. If $x \sim_I y$, then $x * z \sim_I y * z$ by Lemma 13, 5).

Assume now that $x \sim_I y$. We want to prove that $z * x \sim_I z * y$. It is enough to prove $z * x \leq_I z * y$. We have $z * y \leq y \leq_I x$ and therefore, by 1) and 2) above: $z * y \leq_I x$. By Definition 11, item 4), $(z * y) * x \sim I$ $x * (z * y)$. By Lemma 13, part 7) $\neg(z * y) * x \sim_I x * \neg(z * y)$. By Definition 11 $(z * x) * \neg(z * y) \sim_I z * (x * \neg(z * y))$. But $x \leq_I y$ and therefore, by part 5) above, $x * \neg(z * y) \leq_I y * \neg(z * y)$. By F4 $y * \neg(z * y) \leq \neg z$ and therefore $z * (x * \neg(z * y)) = 0$. We see that $(z * x) * \neg(z * y) \sim_I 0$. We conclude that $(z * x) * \neg(z * y) \in I$ and $(z * x) \leq_I z * y$. ▮

Note that the converse of Lemma 12 holds. Any ideal is the kernel of some homomorphism.

Theorem 10 (The homomorphism theorem) *If I is an ideal, then it is the kernel of some homomorphism f that is onto.*

Proof: By Theorem 9, the relation $\sim_I$ is a congruence relation. The operations $\neg$ and $*$ may therefore be defined on the set of equivalence classes under $\sim_I$ in the natural way and f defined by $f(x) = \bar{x}$ is a homomorphisms ($\bar{x}$ is the equivalence class of x under $\sim_I$). One easily sees that the kernel of f is I. ▮

8 Future Work

Here is a list of open questions and lines of enquiry.

- Are the properties of Theorems 1, 3 and 6 independent?
- Do they characterize Hilbert Space Quantum Logic?
- Find other structures that define NCNAB-algebras.

- Find representation theorems for NCNAB-algebras, generalizing known such results for Boolean algebras.

- Consider operations that can be defined using $\neg$ and $*$. For example, $\neg((x * \neg y) * (\neg x * y))$ seems to provide a commutative exclusive disjunction.

- Consider introducing additional operations in the syntax. For example an implication that would be material implication in Boolean algebras and Sasaki hook in Hilbert space satisfying $z \leq x \rightarrow y$ iff $z * x \leq y$, or a disjunction satisfying $z * (x \vee y) \leq w$ iff $z * x \leq w$ and $z * y \leq w$.

- What is the *right* definition of morphisms between P-families?

- Do those morphisms preserve the lattice structure of the underlying Hilbert spaces?

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